# Intertwinning wavelets on graphs: a tool for the inverse problem in E/MEG ?

Joint work with
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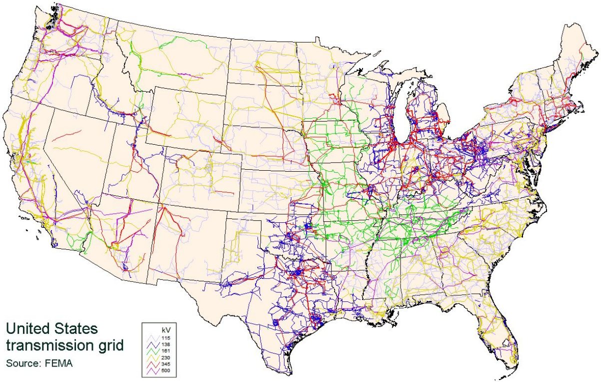
$(-\mathcal{L})$ is a positive symetric matrix with eigenvalues :

$$\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}.$$

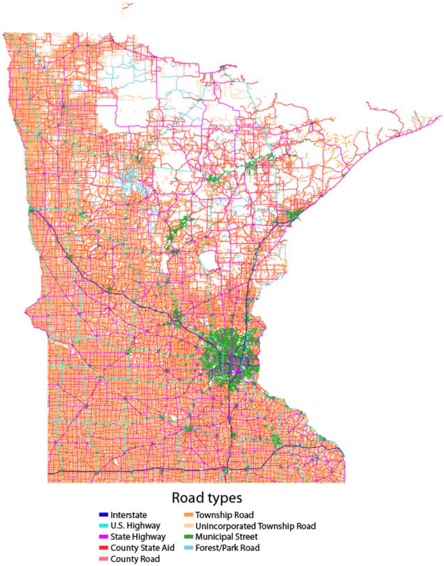
# Examples

## Electrical grid



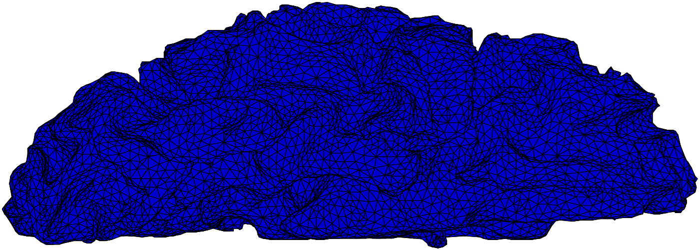
# Examples

## Street network



# Examples

Discretized surfaces (data J. Lefèvre, LIS Marseille)



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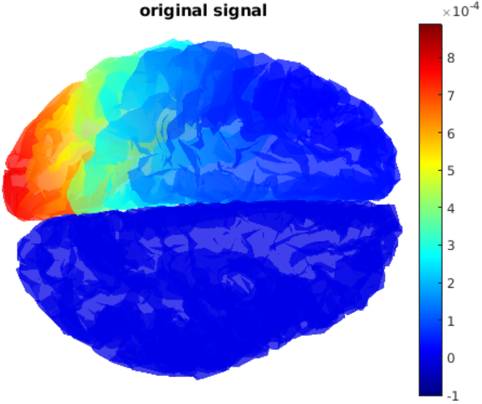

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## Aim.

- We want to build a multiresolution analysis of signals defined on a generic graph, i.e. to encode $f \in \mathbb{R}^n$ as a sum of a general trend, the approximation, and oscillations at different scales, the details :
  our signal $f$ is encoded through $n$ coefficients structured as
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- we would like $[g_1, \cdots, g_k]$ to be a sparse vector whenever $f$ has some "regularity".

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    local "mean" : $\tilde{f}_1(m) = (h \star f_0)(m)$ for the approximation.
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- Iterate.

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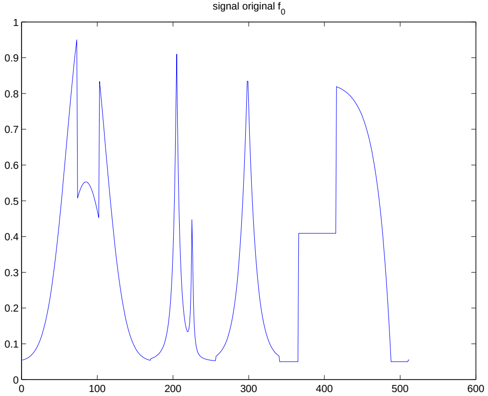
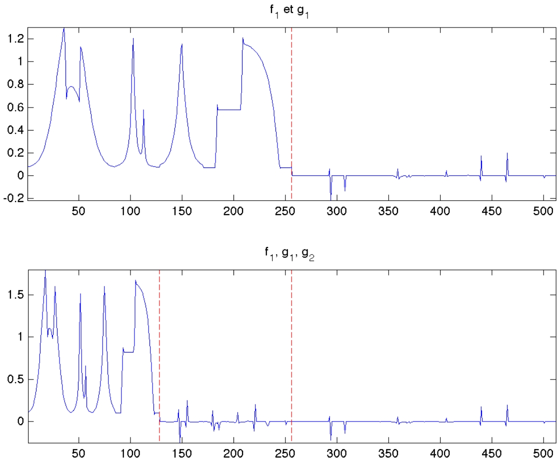
# Example


signal original $f_0$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
100
200
300
400
500
600




one step or two steps of the scheme

## Basis functions

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- Conditioning index, to be able to reconstruct signal from the coefficients.

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- We want $G_1$, $G_2$, $\dots G_k$ to be a sequence of subgraphs which keep as much as possible the important features of the original graph.

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$\rightarrow$ Compute the weights between the points of the subsampling set, which means compute the coefficients of a Laplacian matrix based on the subsampling set.

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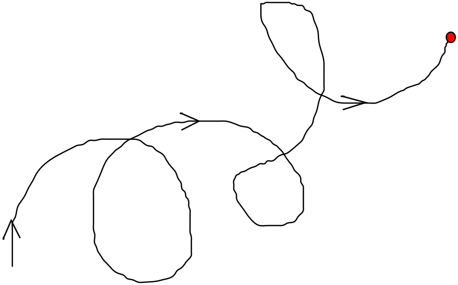
- We denote $X = (X_t, t \geq 0)$ a Markov process with generator $\mathcal{L}$ : $X$ jumps from $x$ to $y$ with probability $w(x,y)/w(x)$ after a random time of law $\mathcal{E}(w(x))$ .

# Subsampling. Choosing a well-spread subset $\bar{\mathcal{X}}$ of $\mathcal{X}$

Wilson's algorithm :



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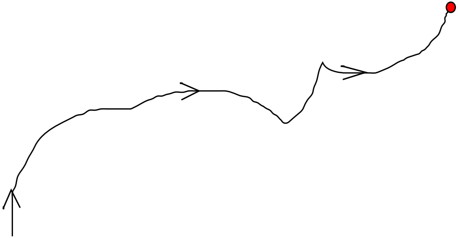

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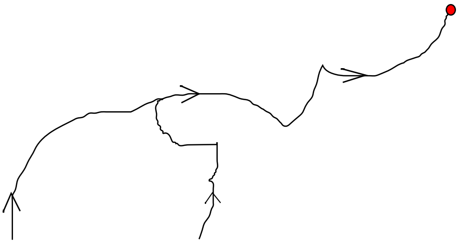

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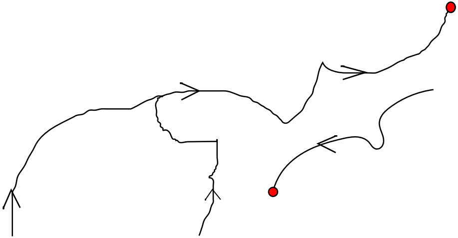

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- Go on until exhaustion of $\mathcal{X}$.

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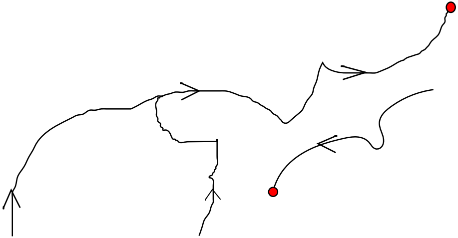

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- Erase the loops.
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# Subsampling. Choosing a well-spread subset $\bar{\mathcal{X}}$ of $\mathcal{X}$

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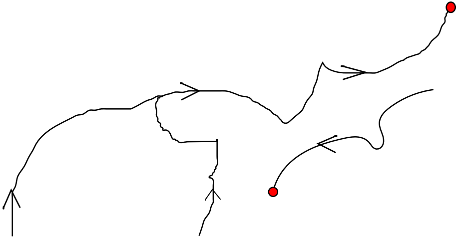

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**Our proposal to choose "one out of two points".**

$$\bar{\mathcal{X}} = \text{set of the trees roots} = \rho(\Phi_q).$$

## Subsampling. Choosing a well-spread subset $\bar{\mathcal{X}}$ of $\mathcal{X}$

### Properties of this set (Wilson (96)).

Let $(X(t), t \geq 0)$ be a Markov process with generator $\mathcal{L}$ and $T_q \sim \mathcal{E}(q)$. Set

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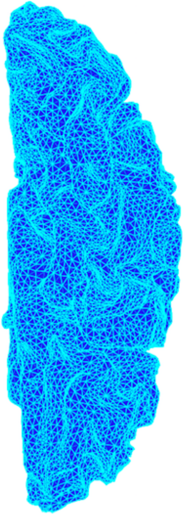
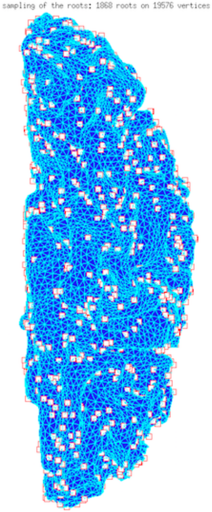
Moreover (Avena & Gaudillière (17)), let $H_{\rho(\Phi_q)}$ the hitting time of $\rho(\Phi_q)$ :
$$\mathbb{E}\left(E_x\left[H_{\rho(\Phi_q)}\right]\right) \text{ does not depend on } x.$$

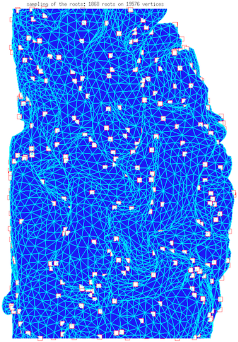
In some sense, the points of $\rho(\Phi_q)$ are well spread in $\mathcal{X}$.

# Examples

Set of roots for small $q$. About 10% of the points are kept.

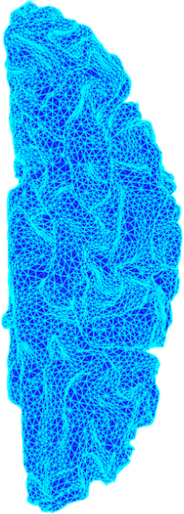


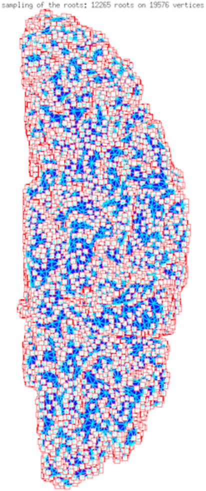


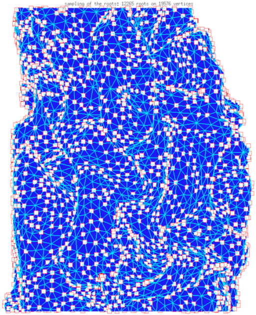



# Examples.

Set of roots for large $q$. About 2/3 of the points are kept.





## Subgraph and filtering.

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- Avoid diagonalization of the Laplacian.

# Weighting and Filtering : Markov intertwining

**Markov intertwining (Rogers & Pitman (81), Diaconis & Fill (90)) :**

A way to link two Markov processes on different state spaces :

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## Why is it useful for us ?

- It provides a natural choice of the weights on $\bar{\mathcal{X}}$ : $\bar{w}(\bar{x}, \bar{y}) = \bar{\mathcal{L}}(\bar{x}, \bar{y})$.
- It provides a natural choice of the approximation coefficients : $\bar{f}(\bar{x}) = \nu_{\bar{x}}(f) = \Lambda f(\bar{x})$.

# Weighting and Filtering : Markov intertwining

Our goal : Given $\bar{\mathcal{X}} \subset \mathcal{X}$, find an approximate solution $(\Lambda, \bar{\mathcal{L}})$ to $\Lambda\mathcal{L} = \bar{\mathcal{L}}\Lambda$ such that

- $\bar{\mathcal{L}}$ is symmetric.
- The $(\nu_{\bar{x}}; \bar{x} \in \bar{\mathcal{X}})$ are "well-localized" in space (non overlapping), to get good reconstruction.
- The $(\nu_{\bar{x}}; \bar{x} \in \bar{\mathcal{X}})$ are "well-localized" in frequency, to separate high and low frequency parts of the signal.
- $\bar{\mathcal{L}}$ and $\Lambda$ are easy to compute (we do not want to compute the spectral decomposition of $\mathcal{L}$).

# Weighting and Filtering : Markov intertwining

**Our proposal.** Assume $\bar{\mathcal{X}} \subset \mathcal{X}$ and $q' > 0$ are given.

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**Definition of the approximation and detail coefficients.**

- For $\bar{x} \in \bar{\mathcal{X}}$, $\bar{f}(\bar{x}) = \nu_{\bar{x}}(f) = K_{q'}f(\bar{x})$.
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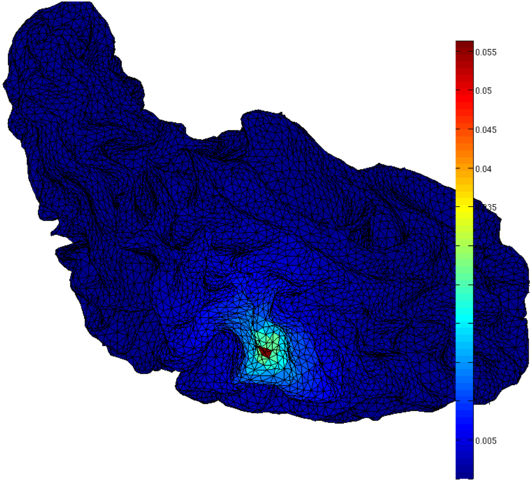
**Some comments.**

- When $q' \ll 1$, $K_{q'}(\bar{x}, \cdot) \simeq \mu$ is well frequency-localized, is a solution to the intertwining relation, is poorly space-localized.
- When $q' \gg 1$, $K_{q'}(\bar{x}, \cdot) \simeq \delta_{\bar{x}}$ is well space-localized. The frequency localization is lost, and depends on the choice of the subset $\bar{\mathcal{X}}$.

# Example of one $\nu_{\bar{x}}$

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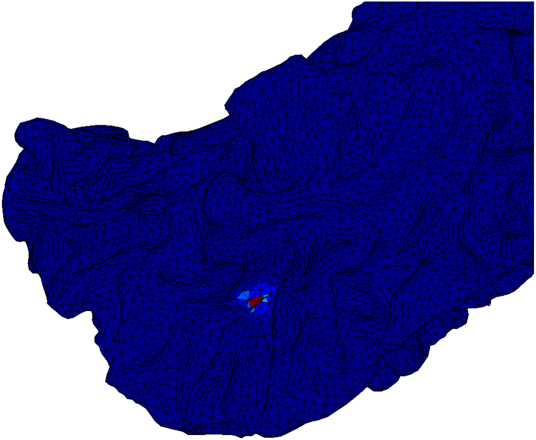

large $q'$

# Some additional results

## An explicit reconstruction formula.

$$f = \begin{pmatrix} \mathrm{Id}_{\bar{\mathcal{X}}} - \frac{1}{q'}\bar{\mathcal{L}} & \mathcal{L}_{\bar{\mathcal{X}}\breve{\mathcal{X}}}(-\mathcal{L}_{\breve{\mathcal{X}}\breve{\mathcal{X}}})^{-1} \\ (-\mathcal{L}_{\breve{\mathcal{X}}\breve{\mathcal{X}}})^{-1}\mathcal{L}_{\breve{\mathcal{X}}\bar{\mathcal{X}}} & q'\mathcal{L}^{-1}_{\breve{\mathcal{X}}\breve{\mathcal{X}}} - \mathrm{Id}_{\breve{\mathcal{X}}} \end{pmatrix} \begin{pmatrix} \bar{f} \\ \breve{f} \end{pmatrix} = \bar{R}\bar{f} + \breve{R}\breve{f}\,,$$

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**Regularity implies small details. Jackson type inequality**

$$\left\|f - \bar{R}_0\bar{R}_1 ... \bar{R}_{K-1}f_K\right\|_{\infty} \leq C_K \left\|\mathcal{L}f\right\|_{\infty} + D_K \left\|f\right\|_{\infty}\ .$$

with

$$\frac{1}{\beta} := \max_{\bar{x}\in\bar{\mathcal{X}}} E_{\bar{x}}\left[H^+_{\breve{\mathcal{X}}} - \tau_1\right],\ \frac{1}{\gamma} := \max_{\breve{x}\in\breve{\mathcal{X}}} E_{\breve{x}}\left[H_{\bar{\mathcal{X}}}\right]\ .$$

# Choice of $q$ and $q'$.

## Wished Properties.

We are looking for $(q, q')$ such that

- $|\rho(\Phi_q)|$ is approximately a given proportion of $|\mathcal{X}|$ (say between [1/2,2/3]) ;
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## A systematic procedure

Let us skip the details...

# Choice of $q$ and $q'$.

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We are looking for $(q, q')$ such that

- $|\rho(\Phi_q)|$ is approximately a given proportion of $|\mathcal{X}|$ (say between [1/2,2/3]) ;
- The reconstruction error is small : $\frac{\bar{\alpha}}{q'}$ and $\frac{q'}{\gamma}$ small.
- The intertwining error is small : $\frac{q'}{\beta}$ small.

## A systematic procedure

Let us skip the details...

- we know how to choose automatically appropriate $(q, q')$ from one scale to the other.

## Choice of $q$ and $q'$.

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- The intertwining error is small : $\frac{q'}{\beta}$ small.

### A systematic procedure

Let us skip the details...

- we know how to choose automatically appropriate $(q, q')$ from one scale to the other.
- and also... we have a procedure to keep the subgraph "sparse"

# The bases and its by by-products

**You give to the algorithm**

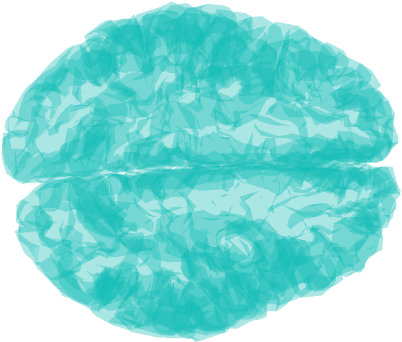
- A graph
- and if you wish a maximum number of levels

**you end up with**

- a sequence of subgraphs
- multiscale bases on your graph until the coarsest approximation level you decided.

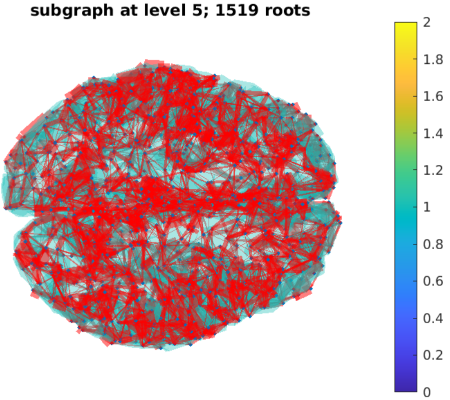
# Subgraphs

(data J.M. Lina, Université de Montréal)



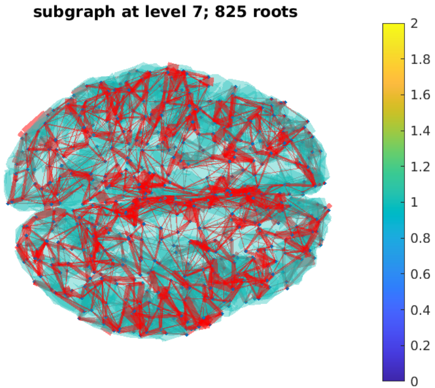
A transparent brain

# Subgraphs



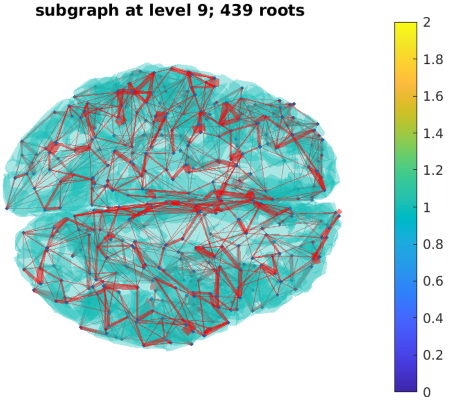

Subgraph at level 5

# Subgraphs



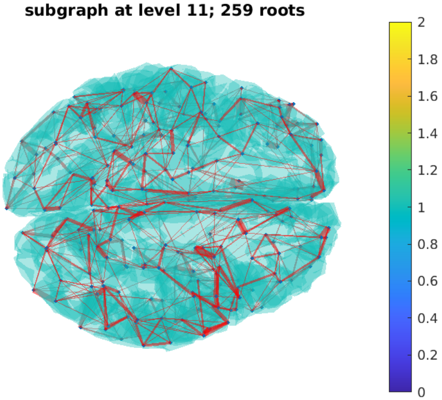

Subgraph at level 7

# Subgraphs



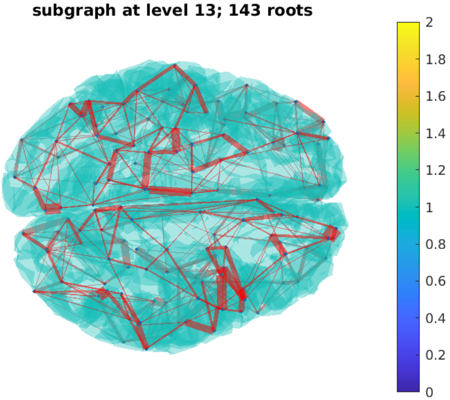

Subgraph at level 9

# Subgraphs



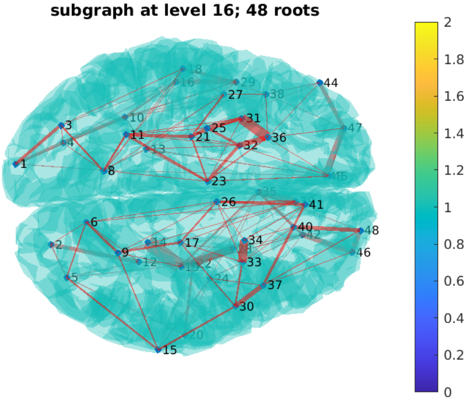

Subgraph at level 11

# Subgraphs




Subgraph at level 13

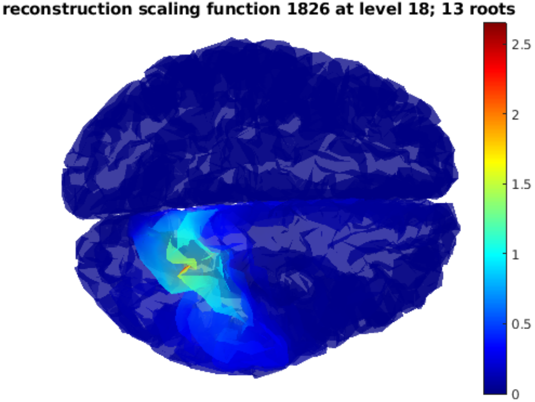
# Subgraphs




Subgraph at level 16

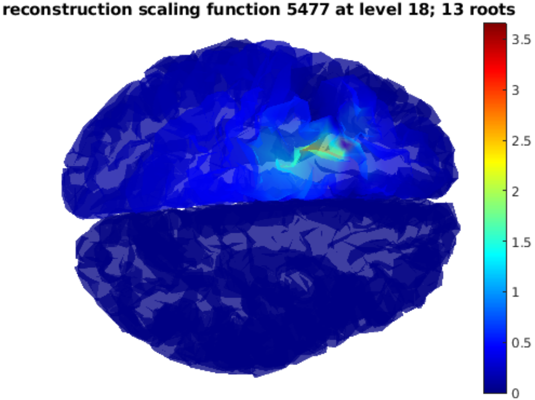
# Choose your level of approximation

Coarse level : approximation function



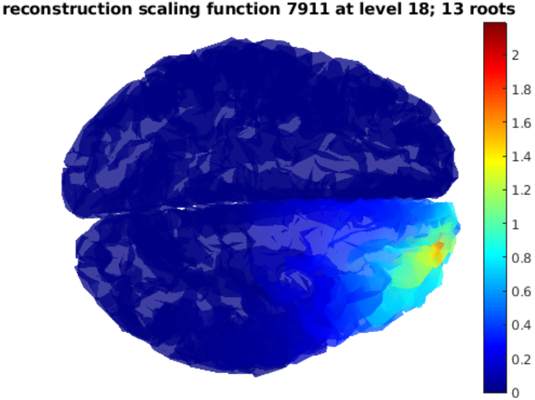
# Choose your level of approximation

Coarse level : approximation function



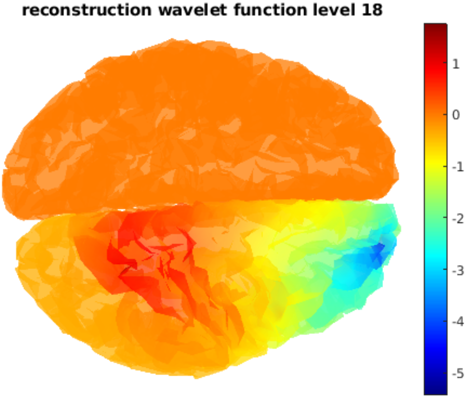
# Choose your level of approximation

Coarse level : approximation function



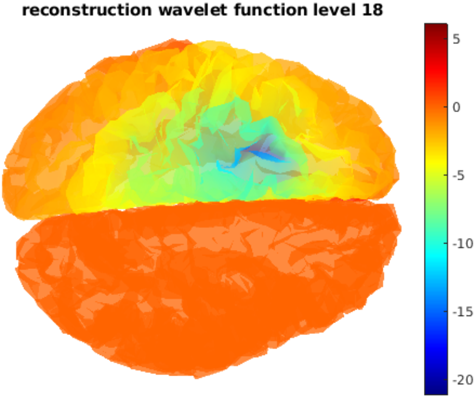
# Choose your level of approximation

Coarse level : wavelet function



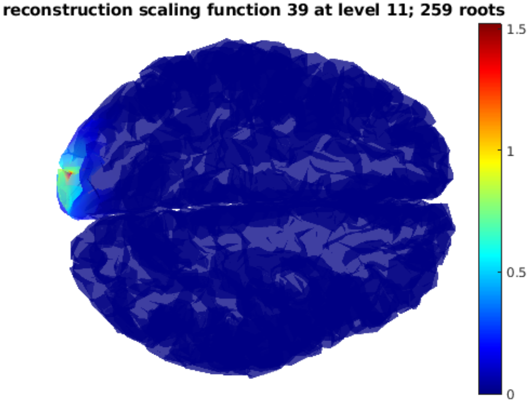
# Choose your level of approximation

Coarse level : wavelet function



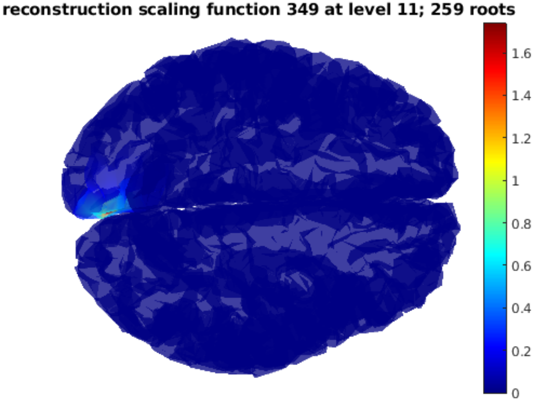
# Choose your level of approximation

Intermediate level : approximation function



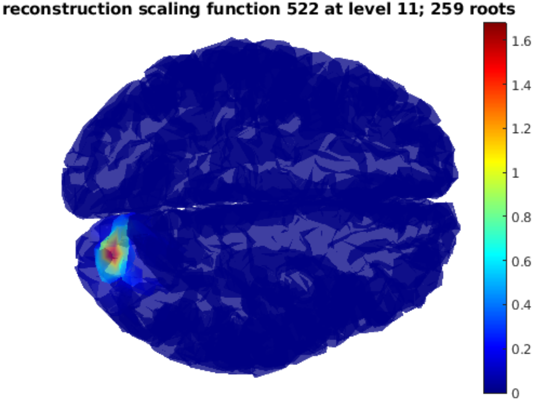
# Choose your level of approximation

Intermediate level : approximation function



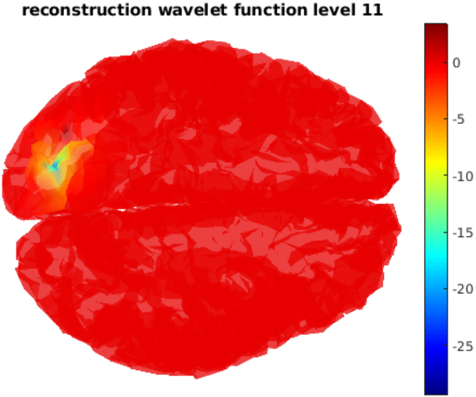
# Choose your level of approximation

Intermdiate level : approximation function



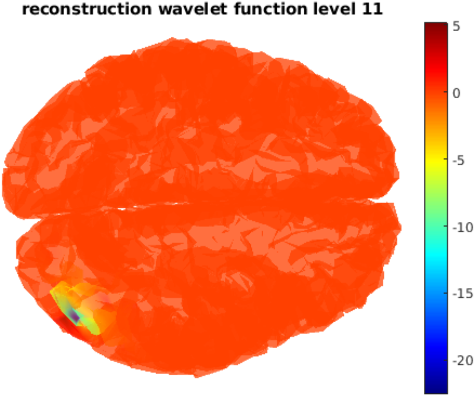
# Choose your level of approximation

Intermediate level : wavelet function



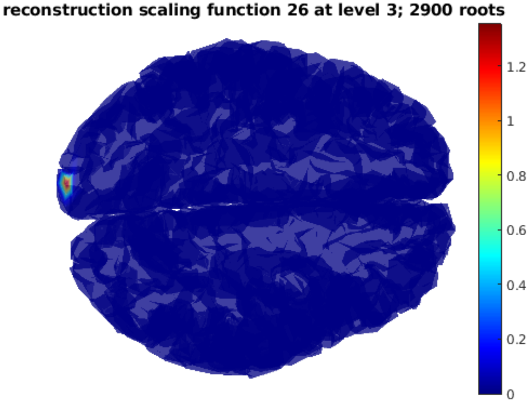
# Choose your level of approximation

Intermediate level : wavelet function



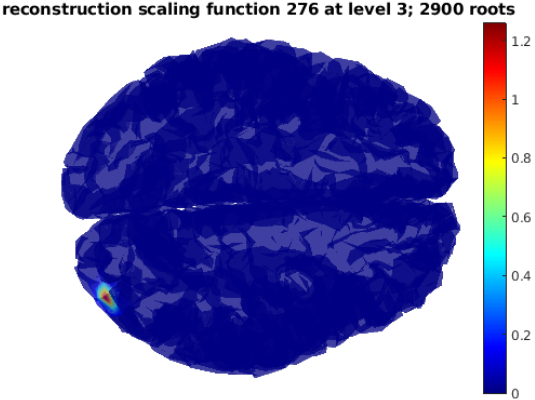
# Choose your level of approximation

Fine level : approximation function



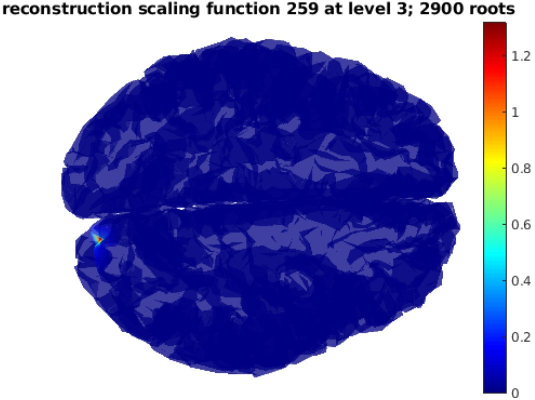
# Choose your level of approximation

Fine level : approximation function



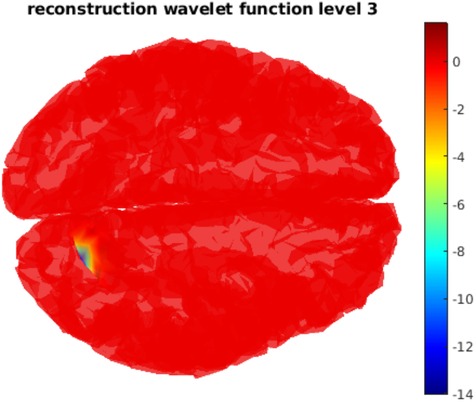
# Choose your level of approximation

Fine level : approximation function



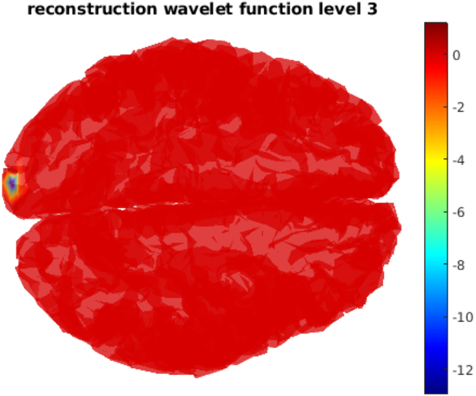
# Choose your level of approximation

Fine level : wavelet function



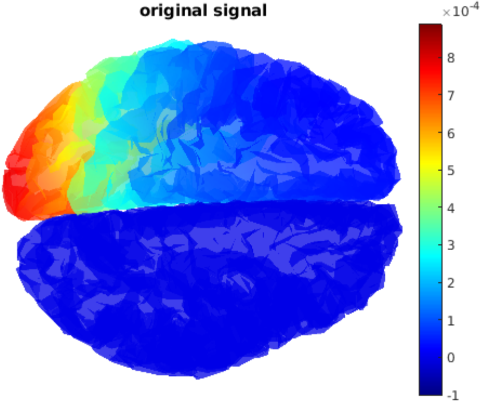
# Choose your level of approximation

Fine level : wavelet function



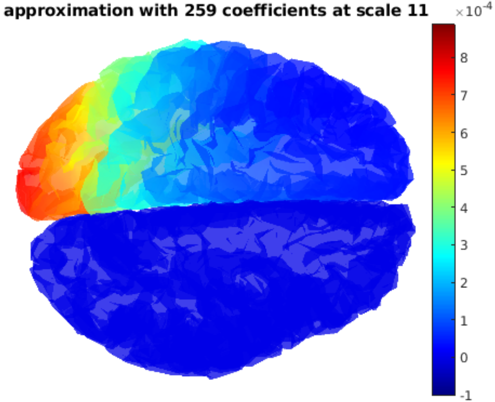
# A sparse representation of smooths signals

A smooth signal



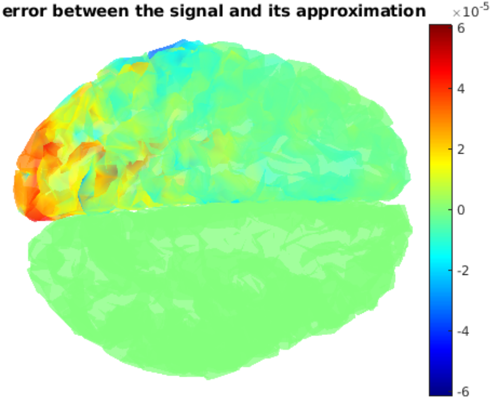
# A sparse representation of smooths signals

Approximation with 3% of coefficients



# A sparse representation of smooths signals

Error of approximation



# Good news !

A Python toolbox should be soon available !